

Action and Entropy of Extreme and Non-Extreme Black Holes

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Abstract

The Hamiltonian actions for extreme and non-extreme black holes are compared and contrasted and a simple derivation of the lack of entropy of extreme black holes is given. In the non-extreme case the wave function of the black hole depends on horizon degrees of freedom which give rise to the entropy. Those additional degrees of freedom are absent in the extreme case.

It has been recently proposed [1], [2] that extreme black holes have zero entropy [3]. The purpose of this note is to adhere to this claim by providing an economical derivation of it. The derivation also helps to set the result in perspective and to relate it to key issues in the quantum theory of gravitation, such as the Wheeler-De Witt equation.

The argument is the application to the case of an extreme black hole of an approach to black hole entropy based on the dimensional continuation of the Gauss-Bonnet theorem developed in [4]. The approach in question had been previously applied to non extreme

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black holes only [5].

To put into evidence as clearly as possible the distinction between extreme and non extreme holes, we first perform the analysis for the non-extreme case and then see how it is modified in the extreme case.

We will deal with gravitation theory in a spacetime of dimension D with positive definite signature (Euclidean formulation). To present the argument in what we believe is its most transparent form for the purpose at hand, we will start with the Hamiltonian action and will only at the end discuss the connection with the Hilbert action.

For non-extreme black holes the Euclidean spacetimes admitted in the action principle have the topology $\mathbb{R}^2 \times S^{D-2}$. It is useful to introduce a polar system of coordinates in the \mathbb{R}^2 factor of $\mathbb{R}^2 \times S^{D-2}$. The reason is that the black hole will have a Killing vector field –the Killing time– whose orbits are circles centered at the horizon. We will take the polar angle in \mathbb{R}^2 as the time variable in a Hamiltonian analysis. An initial surface of time t_1 and a final surface of time t_2 will meet at the origin. There is nothing wrong with the two surfaces intersecting. The Hamiltonian can handle that.

The canonical action

$$I_{can} = \int (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i), \quad (1)$$

without any surface terms added can be taken as the action for the wedge between t_1 and t_2 provided the following quantities are held fixed:

- (i) the intrinsic geometries $^{(D-1)}\mathcal{G}_1$, $^{(D-1)}\mathcal{G}_2$ of the slices $t = t_1$, and $t = t_2$,
- (ii) the intrinsic geometry $^{(D-2)}\mathcal{G}$ of the S^{D-2} at the origin
- (iii) the mass at infinity, with an appropriate asymptotic fall-off for the field.

The term “mass” here refers to the conserved quantity associated with the time Killing vector at infinity. It is thus more general than the P^0 of the Poincaré group, which only exists when the spacetime is asymptotically flat. For example when there is a negative cosmological constant this mass is the value of a generator of the anti-de Sitter group.

Note that we have listed the intrinsic geometry of the S^{D-2} as a variable independent

from the three-geometries of the Slices $t = t_1$ and $t = t_2$. This is because in the variation of the action (1) there is a separate term in the form of an integral over S^{D-2} , which contains the variation of $^{(D-2)}\mathcal{G}$.

It should be observed that there will be no solution of the equations of motion satisfying the given boundary conditions if, for example, one fixes the mass at t_2 to be different from the mass at t_1 . However in the quantum theory one *can* take $M_1 \neq M_2$, the path integral will then yield a factor $\delta(M_2 - M_1)$ in the amplitude. Similarly there will be no solution of the equations of motion unless the geometry of the S^{D-2} at the origin as approached from the slice $t = t_1$, coincides with the one corresponding to $t = t_2$, and unless that common value also coincides with the one taken for the geometry of the S^{D-2} at the origin. However these precautions need not be taken in the path integral, which will automatically enforce them by yielding appropriate δ -functionals. This situation is the same as that arising with the action of a free particle in the momentum representation, where there is no classical solution unless the initial and the final momenta are equal, but yet, one can (and must) compute the amplitude to go from any initial momentum to any final momentum.

To the action (1) one may add any functional of the quantities held fixed and obtain another action appropriate for the same boundary conditions. In particular one may replace (1) by

$$I = I_{can} + B[^{(D-2)}\mathcal{G}], \quad (2)$$

where $B[^{(D-2)}\mathcal{G}]$ is any functional of the $(D-2)$ -geometry at the origin. If we only look at the wedge $t_1 \leq t \leq t_2$ there is no privileged choice for B . However if we demand that the action we adopt should also be appropriate for the complete spacetime, then B is uniquely fixed. This is because when one deals with the complete spacetime the slices $t = t_1$ and $t = t_2$ are identified and neither $^{(D-1)}\mathcal{G}_1$ nor $^{(D-1)}\mathcal{G}_2$ nor $^{(D-2)}\mathcal{G}$ are held fixed. Now, unlike its Minkowskian signature continuation, the Euclidean black hole obeys Einstein's equations *everywhere*. Thus it should be an extremum of the action with only the asymptotic data (mass) held fixed. The demand that the action should be such as to have the black hole as

an extremum with respect to variations of $^{(D-2)}\mathcal{G}$ fixes

$$B = 2\pi A(r_+) \quad (\text{non-extreme case}). \quad (3)$$

where $A(r_+)$ is the area of the S^{D-2} at the origin.

Note that if one includes B for the full spacetime one must include it for the wedge as well. This is because (i) the full spacetime is a particular case of the wedge, and (ii) the boundary term (3) depends only on the $(D-2)$ geometry at the origin and not on t_1 or t_2 ,

The way in which (3) arises is the following. First one writes the metric near the origin in ‘‘Schwarzschild coordinates’’ as

$$ds^2 = N^2(r, x^p)dt^2 + N^{-2}(r, x^p)dr^2 + \gamma_{mn}(r, x^p)dx^m dx^n, \quad (4)$$

with

$$(t_2 - t_1)N^2 = 2\Theta(x^p)(r - r_+) + O(r - r_+)^2 \quad (\text{non-extreme case}). \quad (5)$$

Here r and t are coordinates in \mathbb{R}^2 and x^p are coordinates in S^{D-2} . The parameter Θ is the total proper angle (proper length divided by proper radius) of an arc of very small radius and coordinate angular opening $t_2 - t_1$ in the \mathbb{R}^2 at x^p . For this reason it is called the opening angle. When the sides of the wedge are identified $2\pi - \Theta$ becomes the deficit angle of a conical singularity in \mathbb{R}^2 .

Next, one evaluates the variation of the canonical action (1) to obtain

$$\begin{aligned} \delta I_{can} = & - \int_{S^{(D-2)}(r_+)} \Theta(x^p) \delta \gamma^{1/2}(x^p) d^{D-2}x + \beta \delta M + \\ & \int \pi^{ij} \delta g_{ij} \Big|_1^2 + (\text{terms vanishing on shell}). \end{aligned} \quad (6)$$

Here β is the Killing time separation at infinity.

Last, one observes that when the slices $t = t_1$ and $t = t_2$ are identified, the term $\int \pi^{ij} \delta g_{ij} \Big|_1^2$ cancels out. Thus if M and J are kept fixed but $\gamma^{1/2}(x^p)$ is allowed to vary one must add (3) to (1) in order to obtain from the action principle that at the extremum.

$$\Theta(x^p) = 2\pi \quad (\text{complete spacetime, non-extreme case}). \quad (7)$$

Equation (7) must hold because otherwise there would be a conical singularity at r_+ and Einstein's equations would be violated in the form of a δ -function source at the origin.

Let us now turn to the extreme case. By definition of an extreme black hole the square lapse N^2 has a double root at the origin. Thus one must replace (5) by

$$(t_2 - t_1)N^2 = O(r - r_+)^2 \quad (\text{extreme case}). \quad (8)$$

This means that one must have

$$\Theta(x^p) = 0 \quad (\text{extreme case}), \quad (9)$$

instead of (7). It then follows that

$$B = 0 \quad (\text{extreme case}), \quad (10)$$

so that the canonical action (1) is appropriate as is for extreme black holes.

Note that equation (8) holds not only for the complete spacetime but also for a wedge of the extreme black hole geometry. This implies that (9) must hold also off-shell (for all configurations allowed in the action principle). This is so because for the wedge there is no way to obtain $\Theta = 0$ by extremizing the action since $^{(D-2)}\mathcal{G}$ is held fixed.

The difference between non-extreme and extreme cases has a topological origin. For all Θ 's in the interval

$$0 < \Theta \leq 2\pi, \quad (11)$$

the topology of the t, r piece of the complete spacetime is that of a disk with the boundary at infinity. When $\Theta < 2\pi$ the disk has a conical singularity in the curvature at the origin with deficit angle $2\pi - \Theta$. When $\Theta = 2\pi$ the singularity is absent.

However when $\Theta = 0$ the topology is different. Indeed, what would appear naively to be a source at the origin in the form of a “fully closed cone” –as was misunderstood in [4]– is really the signal of a spacetime with different topology. As the cone closes, its apex recedes to give rise to the infinite throat of an extreme black hole. Thus the origin is effectively

removed from the manifold whose t, r piece is no longer a disk, but rather, an annulus whose inner boundary is at infinite distance.

Now, one wants to include in the action principle fields of a given topology so that one can continuously vary from one to another. Therefore for the complete spacetime of the non-extreme case all fields obeying (11) are allowed so that (7) only holds on-shell. On the other hand, for the extreme case we must have (9) to also hold off-shell. This is so since if the origin is removed, and there is no place to put a conical singularity.

We reach therefore an important conclusion: if we demand that the action should have an extremum on the black hole solution, then we must use a different action for extreme and non-extreme black holes. This means that these two kinds of black holes are to be regarded as drastically different physical objects, much in the same way as particle of however small but finite mass is drastically different from one of zero mass [6]. The discontinuous jump in the action is just the way that the geometrical theory at hand has to remind us that extreme and non-extreme black holes fall into different topological classes.

The action may be rewritten as

$$I = 2\pi\chi A(r_+) + I_{can}, \quad (12)$$

and equations (5) and (8) may be summarized as

$$(t_2 - t_1)N^2 = 2\chi\Theta(x^p)(r - r_+) + O(r - r_+)^2, \quad (13)$$

where χ is the Euler characteristic of the t, r factor of the complete black hole spacetime. For the non-extreme case one has $\chi = 1$ (disk), and for the extreme case $\chi = 0$ (annulus). Expression (12) had been anticipated in [4], where it emerged naturally from a study of the dimensional continuation of the Gauss-Bonnet theorem, but it was missed there that $\chi = 0$ corresponds to extreme black holes.

If one evaluates the action on the black hole solution one finds

$$I_{can}(BlackHole) = 0, \quad (14)$$

because the black hole is stationary ($\dot{g}_{ij} = 0$) and because the constraint equations $\mathcal{H} = \mathcal{H}_i = 0$ hold. Thus one has

$$I(BlackHole) = 2\pi\chi A(r_+). \quad (15)$$

Now, the action (12) is appropriate for keeping M fixed. In statistical thermodynamics this corresponds to the microcanonical ensemble. Thus, for the entropy S in the classical approximation one finds

$$S = (8\pi G\hbar)^{-1} 2\pi\chi A(r_+), \quad (16)$$

where we have restored the universal constants. Thus one sees that extreme black holes ($\chi = 0$) have zero entropy.

A word is now in place about the relation of (12) with the Hilbert action

$$I_H = \frac{1}{2} \int_M \sqrt{g} R d^D x - \int_{\partial M} \sqrt{g} K d^{D-1} x, \quad (17)$$

As was shown in [4], for the complete spacetime (12) and (17) just differ by a boundary term at infinity, which automatically regulates the divergent functional (17). This assertion is not valid for the wedge. In that case, as was also noted in [4], (12) and (17) differ not only by a boundary term at infinity but also by a boundary term at the origin. For the complete spacetime one has

$$I = I_H - B_\infty, \quad (18)$$

whereas for the wedge

$$I = I_H + \pi(2\chi - 1)A(r_+) - B_\infty - \pi A_\infty. \quad (19)$$

For the reasons given above we adopt (12) and not (17) as the action for the wedge.

The discontinuous change in the action between extreme and non-extreme black holes has dramatic consequences for the wave functional of the gravitational field in the presence of a black hole –which one may call for short the wave function of the black hole. Indeed,

in the extreme case, the wave function has the usual arguments, namely, it may be taken to depend on the geometry of the spatial section and on the asymptotic time separation β ,

$$\Psi = \Psi^{[(D-1)]}\mathcal{G}, \beta]. \quad (20)$$

The dependence of Ψ on the three geometry is governed by the Wheeler – De Witt equation

$$\mathcal{H}\Psi = 0, \quad (21)$$

whereas the dependence on the asymptotic time β is governed by the Schrödinger equation

$$\frac{\partial \Psi}{\partial \beta} + M\Psi = 0, \quad (22)$$

where M is the mass as defined by Arnowitt, Deser and Misner (see for example [7]). On the other hand, for the non-extreme case the wave function has an extra argument which may be taken to be the opening angle Θ ,

$$\Psi = \Psi^{[(D-1)]}\mathcal{G}, \beta, \Theta]. \quad (23)$$

Since according to (6) Θ is canonically conjugate to $\gamma^{1/2}$, one has in addition to equations (21) and (22) the extra Schrödinger equation at the horizon [8]

$$\frac{\delta \Psi}{\delta \Theta(x)} - \gamma^{1/2}(x)\Psi = 0. \quad (24)$$

The additional horizon degree of freedom canonical pair $(\gamma^{1/2}, \Theta)$ may be regarded as responsible for the black-hole entropy in the non-extreme case. Indeed there is no entropy in the extreme case precisely because then the origin is absent and there is no place for $(\gamma^{1/2}, \Theta)$ to sit at. This agrees with a point of view previously expressed [8], namely that, –in a way yet to be spelled– the black-hole entropy could be conjectured as arising from “counting conformal factors on the S^{D-2} at r_+ ” or, in terms of the canonically conjugate statement “from counting two-dimensional geometries within a small disk at the horizon”. That disk is removed in the extreme case –and with it the entropy.

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